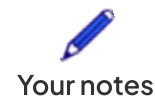


# DP IB Maths: AA HL



## 5.9 Advanced Integration

### Contents

- \* 5.9.1 Integrating Further Functions
- \* 5.9.2 Further Techniques of Integration
- \* 5.9.3 Integrating with Partial Fractions
- \* 5.9.4 Advanced Applications of Integration
- \* 5.9.5 Modelling with Volumes of Revolution

## 5.9.1 Integrating Further Functions



Your notes

As with other problems in integration the results in this revision note may have further uses such as

- evaluating a definite integral
- finding the constant of integration
- finding areas under a curve, between a line and a curve or between two curves



Your notes

## Integrating with Reciprocal Trigonometric Functions

**cosec** (cosecant, csc), **sec** (secant) and **cot** (cotangent) are the reciprocal functions of sine, cosine and tangent respectively.

### What are the antiderivatives involving reciprocal trigonometric functions?

- $\int \sec^2 x \, dx = \tan x + c$
- $\int \sec x \tan x \, dx = \sec x + c$
- $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$
- $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$
- These are **not** given in the **formula booklet** directly
  - they are listed the other way round as 'standard derivatives'
  - be careful with the negatives in the last two results
  - and remember "+c"!

### How do I integrate these if a *linear* function of $x$ is involved?

- All integration rules could apply alongside the results above
- The use of reverse chain rule is particularly common
  - For linear functions the following results can be useful
    - $\int \sec^2(ax + b) \, dx = \frac{1}{a} \tan(ax + b) + c$
    - $\int \sec(ax + b) \tan(ax + b) \, dx = \frac{1}{a} \sec(ax + b) + c$
    - $\int \operatorname{cosec}(ax + b) \cot(ax + b) \, dx = -\frac{1}{a} \operatorname{cosec}(ax + b) + c$
    - $\int \operatorname{cosec}^2(ax + b) \, dx = -\frac{1}{a} \cot(ax + b) + c$
- These are **not** in the formula booklet
  - they can be deduced by spotting reverse chain rule
  - they are not essential to remember but can make problems easier



Your notes

### Examiner Tip

- Even if you think you have remembered these antiderivatives, always use the formula booklet to double check
  - those squares, negatives and "1 over"s are easy to get muddled up!
- Remember to use 'adjust' and 'compensate' for reverse chain rule when coefficients are involved

### Worked example

The graph of  $y = f(x)$  where  $f(x) = \int 2\sec^2 5x \, dx$  passes through the point  $\left(\frac{\pi}{3}, 0\right)$ .

Show that  $5y = 2(\sqrt{3} + \tan 5x)$ .

Reverse chain rule is needed

$$\int 2\sec^2 5x \, dx = 2 \times \frac{1}{5} \int 5\sec^2 5x \, dx$$

↑ 'compensate'
↑ 'adjust'

$$\therefore y = \frac{2}{5} \tan 5x + c$$

$$\int \sec^2 x \, dx = \tan x + c$$

At  $x = \frac{\pi}{3}$ ,  $y = 0$ ,  $0 = \frac{2}{5} \tan \frac{5\pi}{3} + c$

$\frac{5\pi}{3}$  has  $-\sqrt{3}$  written above it.

$$c = \frac{2\sqrt{3}}{5}$$

$$\therefore y = \frac{2}{5} \tan 5x + \frac{2\sqrt{3}}{5}$$

$$y = \frac{2}{5} (\tan 5x + \sqrt{3})$$

$$\therefore 5y = 2(\sqrt{3} + \tan 5x)$$



Your notes

## Integrating with Inverse Trigonometric Functions

**arcsin**, **arccos** and **arctan** are (one-to-one) functions defined as the inverse functions of sine, cosine and tangent respectively.

### What are the antiderivatives involving the inverse trigonometric functions?

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + c$$

$$\int \frac{1}{1+x^2} dx = \arctan x + c$$

- Note that the antiderivative involving **arccos**  $x$  would arise from

$$\int -\frac{1}{\sqrt{1-x^2}} dx = \arccos x + c$$

- However, the negative can be treated as a coefficient of  $-1$  and so

$$\int -\frac{1}{\sqrt{1-x^2}} dx = -\int \frac{1}{\sqrt{1-x^2}} dx = -\arcsin x + c$$

- Similarly,

$$\int \frac{1}{\sqrt{1-x^2}} dx = -\int -\frac{1}{\sqrt{1-x^2}} dx = -\arccos x + c$$

- Unless a question requires otherwise, stick to the first two results
- These are listed in the **formula booklet** the other way round as 'standard derivatives'
- For the antiderivative involving **arctan**  $x$ , note that  $(1+x^2)$  is the same as  $(x^2+1)$

### How do I integrate these expressions if the denominator is not in the correct form?

- Some problems involve integrands that look very **similar** to the above
  - but the denominators start with a number other than one
  - there are three particular cases to consider
- The first two cases involve denominators of the form  $a^2 \pm (bx)^2$  (with or without the square root!)
  - In the case  **$b = 1$**  (i.e. denominator of the form  $a^2 \pm x^2$ ) there are two standard results

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin\left(\frac{x}{a}\right) + c, \quad |x| < a$$

- Both of these **are** given in the **formula booklet**
- Note in the first result,  $a^2 + x^2$  could be written  $x^2 + a^2$

- In cases where  $b \neq 1$  then the integrand can be rewritten by taking a **factor** of  $a^2$ 
  - the factor will be a constant that can be taken outside the integral
  - the remaining denominator will then start with 1
  - e.g.  $9 + 4x^2 = 9\left(1 + \frac{4}{9}x^2\right) = 9\left(1 + \left(\frac{2}{3}x\right)^2\right)$
- The third type of problem occurs when the denominator has a (three term) quadratic
  - i.e. denominators of the form  $ax^2 + bx + c$   
(a rearrangement of this is more likely)
    - the integrand can be rewritten by **completing the square**
    - e.g.  $5 - x^2 + 4x = 5 - (x^2 + 4x) = 5 - [(x + 2)^2 - 4] = 9 - (x + 2)^2$   
This can then be dealt with like the second type of problem above with " $x$ " replaced by " $x + 2$ "
    - This works since the derivative of  $x + 2$  is the same as the derivative of  $x$   
There is essentially no reverse chain rule to consider



Your notes

### Examiner Tip

- Always start integrals involving the inverse trig functions by rewriting the denominator into a recognisable form
  - The numerator and/or any constant factors can be dealt with afterwards, using 'adjust' and 'compensate' if necessary



Your notes

 **Worked example**

a) Find  $\int \frac{1}{9+x^2} dx$ .

The denominator is of the form  $a^2+x^2$  so use the result from the formula booklet: " $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + c$ "

$$\therefore \int \frac{1}{9+x^2} dx = \frac{1}{3} \arctan\left(\frac{x}{3}\right) + c$$

$9=3^2$

b) Find  $\int \frac{1}{\sqrt{5-x^2+4x}} dx$ .



Your notes

The denominator is a three term quadratic so complete the square

$$\begin{aligned}5 - x^2 + 4x &= 5 - [x^2 - 4x] \\ &= 5 - [(x-2)^2 - 4] \\ &= 9 - (x-2)^2\end{aligned}$$

Now write the integral into a recognisable form

$$I = \int \frac{1}{\sqrt{5 - x^2 + 4x}} dx = \int \frac{1}{\sqrt{9 - (x-2)^2}} dx$$

Then use a slight adaption to the result from the formula booklet " $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin\left(\frac{x}{a}\right) + c$ "

$$\therefore I = \arcsin\left(\frac{x-2}{3}\right) + c$$





Your notes

## Integrating Exponential & Logarithmic Functions

Exponential functions have the general form  $y = a^x$ . Special case:  $y = e^x$ .

Logarithmic functions have the general form  $y = \log_a x$ . Special case:  $y = \log_e x = \ln x$ .

### What are the antiderivatives of exponential and logarithmic functions?

- Those involving the special cases have been met before

- $\int e^x dx = e^x + c$

- $\int \frac{1}{x} dx = \ln |x| + c$

- These are given in the **formula booklet**

- Also

- $\int a^x dx = \frac{1}{\ln a} a^x + c$

- This is also given in the **formula booklet**

- By reverse chain rule

- $\int \frac{1}{x \ln a} dx = \log_a |x| + c$

- This is **not** in the formula booklet
    - but the derivative of  $\log_a x$  is given

- There is also the reverse chain rule to look out for

- this occurs when the numerator is (almost) the derivative of the denominator

- $\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c$

### How do I integrate exponentials and logarithms with a *linear* function of x involved?

- For the special cases involving **e** and **ln**

- $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + c$

- $\int \frac{1}{ax+b} dx = \frac{1}{a} \ln |ax+b| + c$

- For the general cases

- $\int a^{px+q} dx = \frac{1}{p \ln a} a^{px+q} + c$

$$\int \frac{1}{(px + q)\ln a} dx = \frac{1}{p} \log_a |px + q| + c$$

- These four results are **not** in the formula booklet but all can be derived using 'adjust and compensate' from **reverse chain rule**

### Examiner Tip

- Remember to always use the modulus signs for logarithmic terms in the antiderivative
  - Once it is deduced that  $g(x)$  in  $\ln |g(x)|$ , say, is guaranteed to be positive, the modulus signs can be replaced with brackets



Your notes



Your notes

 **Worked example**

a) Show that  $\int_1^2 4^x dx = \frac{6}{\ln 2}$ .

From the formula booklet, " $\int a^x dx = \frac{1}{\ln a} a^x + c$ "

$$\therefore \int_1^2 4^x dx = \left[ \frac{1}{\ln 4} 4^x \right]_1^2$$

$$= \frac{16}{\ln 4} - \frac{4}{\ln 4}$$

$$= \frac{12}{\ln 4}$$

$$= \frac{12}{2 \ln 2} \quad \leftarrow \ln 4 = \ln 2^2 = 2 \ln 2$$

$$\therefore \int_1^2 4^x dx = \frac{6}{\ln 2}$$

b) Find  $\int \frac{1}{(2x-1) \ln 3} dx$ .



Your notes

The result  $\int \frac{1}{(px+q) \ln a} dx = \frac{1}{p} \log_a |px+q| + c$

could be used but this is not in the formula booklet.

Alternatively use reverse chain rule with the result "  $f(x) = \log_a x$ ,  $f'(x) = \frac{1}{x \ln a}$  " which is given in the formula booklet!

$$\therefore I = \int \frac{1}{(2x-1) \ln 3} dx = \frac{1}{2} \int \frac{2}{(2x-1) \ln 3} dx$$

← 'adjust'  
↑  
'compensate'

$\therefore I = \frac{1}{2} \log_3 |2x-1| + c$

← ... and '+c'!

↑    ↑  
remember the modulus signs...



Your notes

## 5.9.2 Further Techniques of Integration

### Integration by Substitution

#### What is integration by substitution?

- Integration by substitution is used when an integrand where reverse chain rule is either not obvious or is not spotted
  - in the latter case it is like a “back-up” method for reverse chain rule

#### How do I use integration by substitution?

- For instances where the substitution is not obvious it will be given in a question
  - e.g. Find  $\int \cot x \, dx$  using the substitution  $u = \sin x$
- Substitutions are usually of the form  $u = g(x)$ 
  - in some cases  $u^2 = g(x)$  and other variations are more convenient
    - as these would not be obvious, they would be given in a question
  - if need be, this can be rearranged to find  $x$  in terms of  $u$
- Integration by substitution then involves rewriting the integral, including “ $dx$ ” in terms of  $u$ 
  - STEP 1  
Name the integral to save rewriting it later  
Identify the given substitution  $u = g(x)$
  - STEP 2  
Find  $\frac{du}{dx}$  and rearrange into the form  $f(u) \, du = g(x) \, dx$  such that (some of) the integral can be rewritten in terms of  $u$
  - STEP 3  
If limits are involved, use  $u = g(x)$  to change them from  $x$  values to  $u$  values
  - STEP 4  
Rewrite the integral so everything is in terms of  $u$  rather than  $x$   
This is the step when it may become apparent that  $x$  is needed in terms of  $u$
  - STEP 5  
Integrate with respect to  $u$  and either rewrite in terms of  $x$  or apply the limits using their  $u$  values
- For quotients the substitution usually involves the denominator
- It may be necessary to use ‘adjust and compensate’ to deal with any coefficients in the integrand

- Although  $\frac{du}{dx}$  can be treated like a fraction it should be appreciated that this is a 'shortcut' and the maths behind it is beyond the scope of the IB course

### Examiner Tip

- If a substitution is not given in a question, it is usually because it is obvious
  - If you can't see anything obvious, or you find that your choice of substitution doesn't reduce the integrand to something easy to integrate, consider that it may not be a substitution question



Your notes



Your notes

### Worked example

Use the substitution  $u = (1 + 2x)$  to evaluate  $\int_0^1 x(1 + 2x)^7 dx$ .

STEP 1: Name the integral, identify the substitution

$$I = \int_0^1 x(1 + 2x)^7 dx$$

$$u = 1 + 2x$$

STEP 2: Find  $\frac{du}{dx}$  and rearrange

$$\frac{du}{dx} = 2$$

$$\frac{1}{2} du = dx$$

STEP 3: Change limits from  $x$  values to  $u$  values

$$x = 0, \quad u = 1 + 2(0) = 1$$

$$x = 1, \quad u = 1 + 2(1) = 3$$

STEP 4: Rewrite the integral, find  $x$  in terms of  $u$

$$I = \int_1^3 \frac{1}{2}(u-1)u^7 \times \frac{1}{2} du = \frac{1}{4} \int_1^3 (u^8 - u^7) du$$

$$\begin{aligned} &\uparrow \\ &x \text{ in terms of } u \\ &u = 1 + 2x \\ &\therefore x = \frac{1}{2}(u-1) \end{aligned}$$

STEP 5: Integrate and evaluate

$$I = \frac{1}{4} \left[ \frac{u^9}{9} - \frac{u^8}{8} \right]_1^3$$

$$I = \frac{1}{4} \left[ \left( \frac{3^9}{9} - \frac{3^8}{8} \right) - \left( \frac{1^9}{9} - \frac{1^8}{8} \right) \right]$$

$$\therefore I = \frac{6151}{18}$$



Your notes

## Integration by Parts

### What is integration by parts?

- Integration by parts is generally used to integrate the product of two functions
  - however reverse chain rule and/or substitution should be considered first
    - e.g.  $\int 2x \cos(x^2) dx$  can be solved using reverse chain rule or the substitution  $u = x^2$
- Integration by parts is essentially 'reverse product rule'
  - whilst every product can be differentiated, not every product can be integrated (analytically)

### What is the formula for integration by parts?

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

- This is given in the **formula booklet** alongside its alternative form  $\int u dv = uv - \int v du$

### How do I use integration by parts?

- For a given integral  $u$  and  $\frac{dv}{dx}$  (rather than  $u$  and  $v$ ) are assigned functions of  $x$
- Generally, the function that becomes simpler when differentiated should be assigned to  $u$
- There are various stages of integrating in this method
  - only one overall constant of integration (" +c ") is required
  - put this in at the last stage of working
  - if it is a definite integral then " +c " is not required at all

STEP 1

Name the integral if it doesn't have one already!

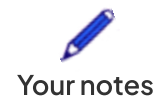
This saves having to rewrite it several times - I is often used for this purpose.

$$\text{e.g. } I = \int x \sin x dx$$

STEP 2

Assign  $u$  and  $\frac{dv}{dx}$ .





Differentiate  $u$  to find  $\frac{du}{dx}$  and integrate  $\frac{dv}{dx}$  to find  $v$

$$u = x \quad v = -\cos x$$

e.g.  $\frac{du}{dx} = 1 \quad \frac{dv}{dx} = \sin x$

STEP 3

Apply the integration by parts formula

e.g.  $I = -x \cos x - \int -\cos x \, dx$

STEP 4

Work out the second integral,  $\int v \frac{du}{dx} \, dx$

Now include a "+c" (unless definite integration)

e.g.  $I = -x \cos x + \sin x + c$

STEP 5

Simplify the answer if possible or apply the limits for definite integration

e.g.  $I = \sin x - x \cos x + c$

- In trickier problems other rules of differentiation and integration may be needed
  - chain, product or quotient rule
  - reverse chain rule, substitution

### Can integration by parts be used when there is only a single function?

- Some single functions (non-products) are awkward to integrate directly
  - e.g.  $y = \ln x$ ,  $y = \arcsin x$ ,  $y = \arccos x$ ,  $y = \arctan x$
- These can be integrated using parts however
  - rewrite as the product ' $1 \times f(x)$ ' and choose  $u = f(x)$  and  $\frac{dv}{dx} = 1$
  - 1 is easy to integrate and the functions above have standard derivatives listed in the formula booklet

### Examiner Tip

- If  $\ln x$  or one of the inverse trig functions are one of the functions involved in the product then these should be assigned to " $u$ " when applying parts
  - They are (relatively) easy to differentiate (to find  $u'$ ) but are awkward to integrate



Your notes



Your notes

 **Worked example**

a) Find  $\int 5xe^{3x} dx$ .

STEP 1: Name the integral

$$I = \int 5xe^{3x} dx = 5 \int xe^{3x} dx$$

STEP 2: Assign  $u$  and  $v'$

Find  $u'$  and  $v$

$$u = x \quad v = \frac{1}{3}e^{3x} \text{ (reverse chain rule)}$$

$$u' = 1 \quad v' = e^{3x}$$

$x$  becomes simpler when differentiated

STEP 3: Apply the integration by parts formula

$$I = 5 \left[ \frac{1}{3}xe^{3x} - \int \frac{1}{3}e^{3x} dx \right]$$

STEP 4: Work out the second integral

$$I = 5 \left[ \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c \right] \leftarrow \text{include "+c" at last working stage}$$

$\nwarrow$  reverse chain rule

STEP 5: Simplify

$$I = \frac{5}{9}e^{3x}(3x-1) + c$$

b) Show that  $\int 8x \ln x dx = 2x^2(1 + \ln x^2) + c$ .



Your notes

STEP 1: Name the integral

$$I = \int 8x \ln x \, dx$$

STEP 2: Assign  $u$  and  $v$  - as  $\ln$  is involved,  $u = \ln x$ Find  $u'$  and  $v$ 

$$u = \ln x \quad v = 4x^2$$

$$u' = \frac{1}{x} \quad v' = 8x$$

STEP 3: Apply the integration by parts formula

$$I = 4x^2 \ln x - \int 4x^2 \times \frac{1}{x} \, dx = 4x^2 \ln x - \int 4x \, dx$$

STEP 4: Work out the second integral, include "+c" at this stage

$$I = 4x^2 \ln x - 2x^2 + c$$

STEP 5: Simplify

$$I = 2x^2(2 \ln x - 1) + c$$

$$\therefore I = 2x^2(\ln x^2 - 1) + c$$



Your notes

## Repeated Integration by Parts

### When will I have to repeat integration by parts?

- In some problems, applying integration by parts still leaves the second integral as a product of two functions of  $x$ 
  - integration by parts will need to be applied again to the second integral
- This occurs when one of the functions takes more than one derivative to become simple enough to make the second integral straightforward
  - These functions usually have the form  $x^2g(x)$

### How do I apply integration by parts more than once?

STEP 1

Name the integral if it doesn't have one already!

STEP 2

Assign  $u$  and  $\frac{dv}{dx}$ . Find  $\frac{du}{dx}$  and  $v$

STEP 3

Apply the integration by parts formula

STEP 4

Repeat STEPS 2 and 3 for the second integral

STEP 5

Work out the second integral and include a "+c" if necessary

STEP 6

Simplify the answer or apply limits

### What if neither function ever becomes simpler when differentiating?

- It is possible that integration by parts will end up in a seemingly endless loop
  - consider the product  $e^x \sin x$
  - the derivative of  $e^x$  is  $e^x$ 
    - no matter how many times a function involving  $e^x$  is differentiated, it will still involve  $e^x$
  - the derivative of  $\sin x$  is  $\cos x$ 
    - $\cos x$  would then have derivative  $-\sin x$ , and so on
    - no matter how many times a function involving  $\sin x$  or  $\cos x$  is differentiated, it will still involve  $\sin x$  or  $\cos x$
- This loop can be trapped by spotting when the second integral becomes identical to (or a multiple of) the original integral
  - naming the original integral ( $I$ ) at the start helps

- $I$  then appears twice in integration by parts
  - e.g.  $I = g(x) - I$   
where  $g(x)$  are parts of the integral not requiring further work
- It is then straightforward to rearrange and solve the problem
  - e.g.  $2I = g(x) + c$   
$$I = \frac{1}{2}g(x) + c$$



Your notes



Your notes

 **Worked example**

a) Find  $\int x^2 \cos x \, dx$ .

STEP 1: Name the integral

$$I = \int x^2 \cos x \, dx$$

STEP 2: Assign  $u$  and  $v'$

Find  $u'$  and  $v$

$$\begin{array}{ll} u = x^2 & v = \sin x \\ u' = 2x & v' = \cos x \end{array}$$

$x^2$  becomes 'simpler' when differentiated

STEP 3: Apply the integration by parts formula

$$I = x^2 \sin x - 2 \int x \sin x \, dx$$

STEP 4: Repeat STEPS 2 and 3 for the second integral

$$\begin{array}{ll} u = x & v = -\cos x \\ u' = 1 & v' = \sin x \end{array}$$

$$I = x^2 \sin x - 2 \left[ -x \cos x - \int -\cos x \, dx \right]$$

STEP 5: Work out the second integral now it is straightforward

$$I = x^2 \sin x + 2x \cos x - 2 \sin x + c$$

STEP 6: Simplify

$$I = (x^2 - 2) \sin x + 2x \cos x + c$$

b) Find  $\int e^x \sin x \, dx$ .



Your notes

STEP 1: Name the integral

$$I = \int e^x \sin x \, dx$$

STEP 2: Assign  $u$  and  $v$ . Neither function becomes simpler when differentiated. Find  $u'$  and  $v$ .

$$u = e^x \quad v = -\cos x$$

$$u' = e^x \quad v' = \sin x$$

STEP 3: Apply the integration by parts formula

$$I = -e^x \cos x - \int -e^x \cos x \, dx = -e^x \cos x + \int e^x \cos x \, dx$$

STEP 4: Repeat STEPS 2 and 3 for the second integral

$$u = e^x \quad v = \sin x$$

$$u' = e^x \quad v' = \cos x$$

$$I = -e^x \cos x + \left[ e^x \sin x - \int e^x \sin x \, dx \right]$$

Spot that this is the same as the original question, i.e.  $I$

STEP 5: "Work out" the second integral, include "+c" at this stage

$$I = e^x \sin x - e^x \cos x - I + c$$

STEP 6: Simplify

$$2I = e^x (\sin x - \cos x) + c$$

$$\therefore I = \frac{1}{2} e^x (\sin x - \cos x) + c_1 \quad (\text{where } c_1 = \frac{1}{2} c)$$





Your notes

## 5.9.3 Integrating with Partial Fractions

### Integrating with Partial Fractions

#### What are partial fractions?

- Partial fractions arise when a quotient is rewritten as the **sum** of fractions
  - The process is the opposite of adding or subtracting fractions
- Each partial fraction has a denominator which is a **linear factor** of the quotient's denominator
  - e.g. A quotient with a denominator of  $x^2 + 4x + 3$ 
    - factorises to  $(x + 1)(x + 3)$
    - so the quotient will split into two partial fractions
    - one with the (linear) denominator  $(x + 1)$
    - one with the (linear) denominator  $(x + 3)$

#### How do I know when to use partial fractions in integration?

- For this course, the denominators of the quotient will be of quadratic form
  - i.e.  $f(x) = ax^2 + bx + c$
  - check to see if the quotient can be written in the form  $\frac{f'(x)}{f(x)}$ 
    - in this case, reverse chain rule applies
- If the denominator does not factorise then the **inverse trigonometric functions** are involved

#### How do I integrate using partial fractions?

STEP 1

Write the quotient in the integrand as the sum of partial fractions

This involves factorising the denominator, writing it as an identity of two partial fractions and using values of  $x$  to find their numerators

$$\text{e.g. } I = \int \frac{1}{x^2 + 4x + 3} dx = \int \frac{1}{(x + 1)(x + 3)} dx = \frac{1}{2} \int \left( \frac{1}{x + 1} - \frac{1}{x + 3} \right) dx$$

STEP 2

Integrate each partial fraction leading to an expression involving the sum of natural logarithms

$$\text{e.g. } I = \frac{1}{2} \int \left( \frac{1}{x + 1} - \frac{1}{x + 3} \right) dx = \frac{1}{2} [\ln |x + 1| - \ln |x + 3|] + c$$

STEP 3

Use the laws of logarithms to simplify the expression and/or apply the limits

(Simplifying first may make applying the limits easier)

$$\text{e.g. } I = \frac{1}{2} \ln \left| \frac{x+1}{x+3} \right| + c$$

- By rewriting the constant of integration as a logarithm ( $c = \ln k$ , say) it is possible to write the final answer as a single term

$$\text{e.g. } I = \frac{1}{2} \ln \left| \frac{x+1}{x+3} \right| + \ln k = \ln \sqrt{\left| \frac{x+1}{x+3} \right|} + \ln k = \ln \left( k \sqrt{\left| \frac{x+1}{x+3} \right|} \right)$$

### Examiner Tip

- Always check to see if the numerator can be written as the derivative of the denominator
  - If so then it is reverse chain rule, not partial fractions
  - Use the number of marks a question is worth to help judge how much work should be involved



Your notes



Your notes

 **Worked example**

Find  $\int \frac{3x+1}{x^2+3x-10} dx$ .

The integrand is NOT of the form  $\frac{f'(x)}{f(x)}$  but the denominator does factorise

STEP 1: Write the quotient as partial fractions

$$\frac{3x+1}{x^2+3x-10} \equiv \frac{A}{x+5} + \frac{B}{x-2}$$

$$3x+1 \equiv A(x-2) + B(x+5)$$

$$\text{Let } x=2, \quad 7=7B, \quad B=1$$

$$\text{Let } x=-5, \quad -14=-7A, \quad A=2$$

$$\therefore I = \int \frac{3x+1}{x^2+3x-10} dx = \int \left( \frac{2}{x+5} + \frac{1}{x-2} \right) dx$$

STEP 2: Integrate the partial fractions

$$I = 2 \ln|x+5| + \ln|x-2| + c$$

STEP 3: Simplify using laws of logarithms

$$I = \ln(x+5)^2 + \ln|x-2| + c$$

$$\therefore I = \ln |(x+5)^2(x-2)| + c$$

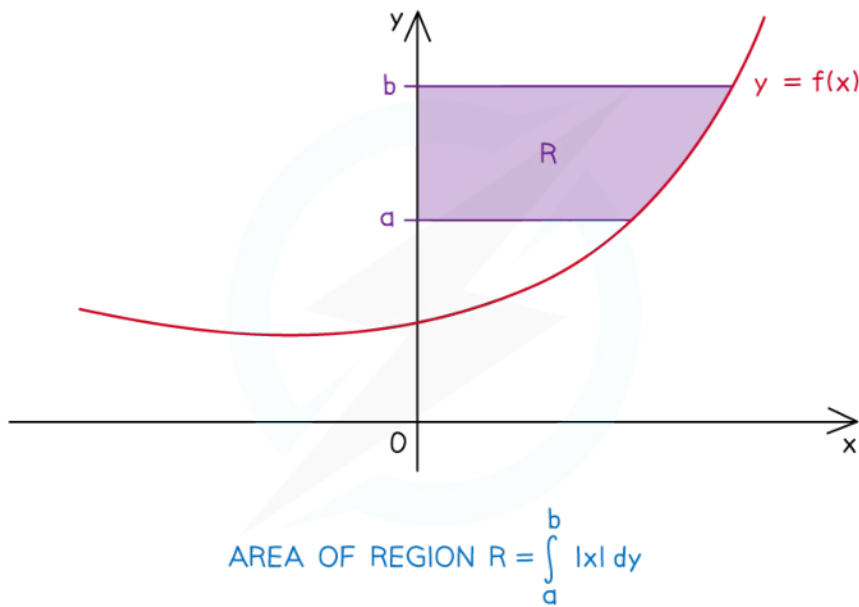


Your notes

## 5.9.4 Advanced Applications of Integration

### Area Between Curve & y-axis

What is meant by the area between a curve and the y-axis?



Copyright © Save My Exams. All Rights Reserved

- The area referred to is the region bounded by
  - the graph of  $y = f(x)$
  - the y-axis
  - the horizontal line  $y = a$
  - the horizontal line  $y = b$
- The exact area can be found by evaluating a definite integral
- The graph of  $y = f(x)$  could be a straight line
  - using basic shape area formulae may be easier than integration
    - e.g. area of a trapezium:  $A = \frac{1}{2}h(a + b)$

**How do I find the area between a curve and the y-axis?**

- Use the formula

$$A = \int_a^b |x| \, dy$$



Your notes

- This is given in the **formula booklet**
- The function is normally given in the form  $y = f(x)$ 
  - so will need rearranging into the form  $x = g(y)$
- $a$  and  $b$  may not be given directly as could involve the  $x$ -axis ( $y = 0$ ) and/or a root of  $x = g(y)$ 
  - use a GDC to plot the curve, sketch it and highlight the area to help

STEP 1

Identify the limits  $a$  and  $b$ 

Sketch the graph of  $y = f(x)$  or use a GDC to do so, especially if  $a$  and  $b$  are not given directly in the question

STEP 2

Rearrange  $y = f(x)$  into the form  $x = g(y)$ This is similar to finding the inverse function  $f^{-1}(x)$ 

STEP 3

Evaluate the formula to evaluate the integral and find the area required

If using a GDC remember to include the modulus ( $| \dots |$ ) symbols around  $x$ 

- In trickier problems some (or all) of the area may be 'negative'
  - this will be any area that is left of the  $y$ -axis (negative  $x$ -values)
  - $|x|$  makes such areas 'positive'
    - a GDC will apply ' $|x|$ ' automatically as long as the  $| \dots |$  are included
    - otherwise, to apply ' $|x|$ ', split the integral into positive and negative parts; write an integral and evaluate each part separately and add the modulus of each part together to give the total area

### Examiner Tip

- Sketch and/or use your GDC to help visualise what the problem looks like



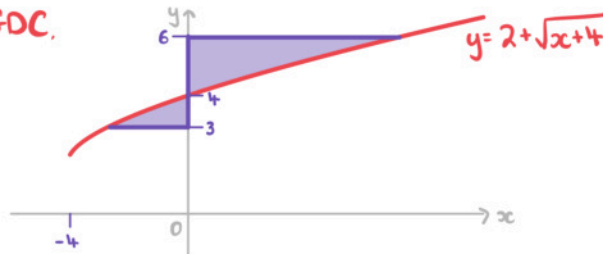
Your notes

### Worked example

Find the area enclosed by the curve with equation  $y = 2 + \sqrt{x+4}$ , the  $y$ -axis and the horizontal lines with equations  $y = 3$  and  $y = 6$ .

STEP 1: Identify limits, sketch graph/use GOC

From GOC,



STEP 2: Rearrange  $y = f(x)$  into  $x = g(y)$

$$y = 2 + \sqrt{x+4}$$

$$x = (y-2)^2 - 4 = y^2 - 4y + 4 - 4$$

$$x = y^2 - 4y$$

STEP 3: Evaluate integral to find area

As some area 'is' negative, split the integral

$$A = - \int_3^4 (y^2 - 4y) dy + \int_4^6 (y^2 - 4y) dy$$

↑  
From GOC/sketch

this area is 'negative'

If using GOC you can

still do this in one go:

$$\int_3^6 |y^2 - 4y| dy$$

$$\therefore A = \left[ \frac{y^3}{3} - 2y^2 \right]_4^6 - \left[ \frac{y^3}{3} - 2y^2 \right]_3^4$$

$$A = \left[ (72 - 72) - \left( \frac{64}{3} - 32 \right) \right] - \left[ \left( \frac{64}{3} - 32 \right) - (9 - 18) \right]$$

$$A = \frac{32}{3} - - \frac{5}{3}$$

$$\therefore A = \frac{37}{3} \text{ square units}$$



Your notes

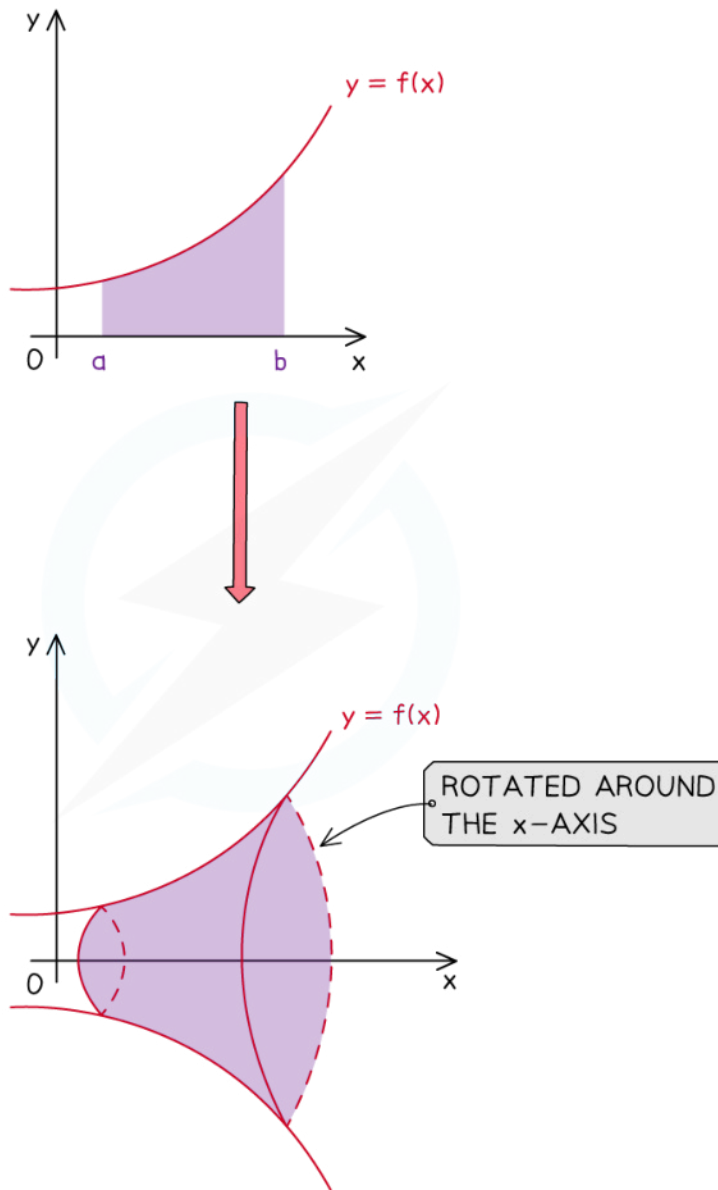


Your notes

## Volumes of Revolution Around x-axis

### What is a volume of revolution around the x-axis?

- A **solid of revolution** is formed when an **area** bounded by a function  $y = f(x)$  (and other boundary equations) is rotated  $2\pi$  radians ( $360^\circ$ ) around the  $x$ -axis
- The **volume of revolution** is the volume of this solid



Copyright © Save My Exams. All Rights Reserved





Your notes

- Be careful – the ‘front’ and ‘back’ of this solid are flat
  - they were created from straight (vertical) lines
  - 3D sketches can be misleading

### How do I solve problems involving the volume of revolution around x-axis?

- Use the formula

$$V = \pi \int_a^b y^2 dx$$

- This is given in the **formula booklet**
- $y$  is a function of  $x$
- $x = a$  and  $x = b$  are the equations of the (vertical) lines bounding the area
  - If  $x = a$  and  $x = b$  are not stated in a question, the boundaries could involve the  $y$ -axis ( $x = 0$ ) and/or a root of  $y = f(x)$
  - Use a GDC to plot the curve, sketch it and highlight the area to help
- Visualising the solid created is helpful
  - Try sketching some functions and their solids of revolution to help

STEP 1

Identify the limits  $a$  and  $b$

Sketching the graph of  $y = f(x)$  or using a GDC to do so is helpful, especially when  $a$  and  $b$  are not given directly in the question

STEP 2

Square  $y$

STEP 3

Use the formula to evaluate the integral and find the volume of revolution

An answer may be required in exact form

#### Examiner Tip

- If the given function involves a square root(s), problems can seem quite daunting
  - However, this is often deliberate, as the square root will be squared when applying the Volume of Revolution formula, and should leave the integrand as something more manageable
- Whether a diagram is given or not, using your GDC to plot the curve, limits, etc (where possible) can help you to visualise and make progress with problems



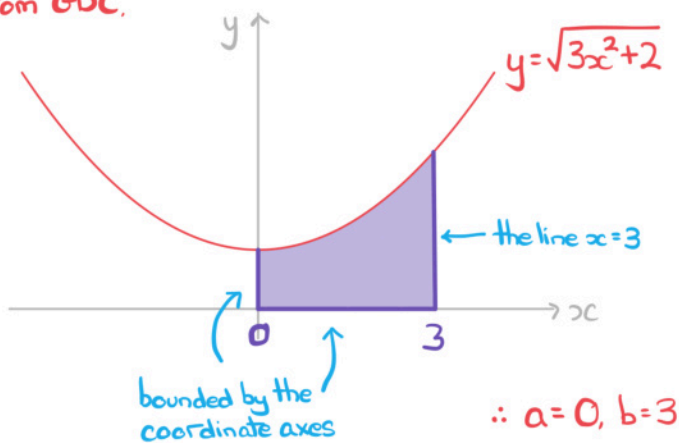
Your notes

**Worked example**

Find the volume of the solid of revolution formed by rotating the region bounded by the graph of  $y = \sqrt{3x^2 + 2}$ , the coordinate axes and the line  $x = 3$  by  $2\pi$  radians around the  $x$ -axis. Give your answer as an exact multiple of  $\pi$ .

STEP 1: Identify limits, sketch graph/use GOC

From GOC,



STEP 2: Square  $y$

$$y^2 = (\sqrt{3x^2 + 2})^2 = 3x^2 + 2$$

STEP 3: Find the volume

$$\begin{aligned} V &= \pi \int_0^3 (3x^2 + 2) \, dx = \pi [x^3 + 2x]_0^3 \\ &= \pi (27 + 6) \end{aligned}$$

$\therefore V = 33\pi \text{ cubic units}$



Your notes

## Volumes of Revolution Around y-axis

### What is a volume of revolution around the y-axis?

- Very similar to above, this is a **solid of revolution** which is formed when an **area** bounded by a function  $y = f(x)$  (and other boundary equations) is rotated  $2\pi$  radians ( $360^\circ$ ) around the  $y$ -axis
- The **volume of revolution** is the volume of this solid

### How do I solve problems involving the volume of revolution around y-axis?

- Use the formula

$$V = \pi \int_a^b x^2 dy$$

- This is given in the **formula booklet**
- The function is usually given in the form  $y = f(x)$ 
  - so will need rearranging into the form  $x = g(y)$
- $a$  and  $b$  may not be given directly as could involve the  $x$ -axis ( $y = 0$ ) and/or a root of  $x = g(y)$ 
  - Use a GDC to plot the curve, sketch it and highlight the area to help
- Visualising the solid created is helpful

STEP 1

Identify the limits  $a$  and  $b$

Sketching the graph of  $y = f(x)$  or using a GDC to do so is helpful, especially if  $a$  and  $b$  are not given directly in the question

STEP 2

Rearrange  $y = f(x)$  into the form  $x = g(y)$

This is similar to finding the inverse function  $f^{-1}(x)$

STEP 3

Square  $x$

STEP 4

Use the formula to evaluate the integral and find the volume of revolution

An answer may be required in exact form

### Examiner Tip

- If the given function involves a square root, problems can seem quite daunting
  - This is often deliberate, as the square root will be squared when applying the Volume of Revolution formula and the integrand will then become more manageable
- Whether a diagram is given or not, using your GDC to plot the curve, limits, etc (where possible) can help you to visualise the problem and make progress

 **Worked example**

Find the volume of the solid of revolution formed by rotating the region bounded by the graph of  $y = \arcsin(2x + 1)$  and the coordinate axes by  $2\pi$  radians around the  $y$ -axis. Give your answer to three significant figures.



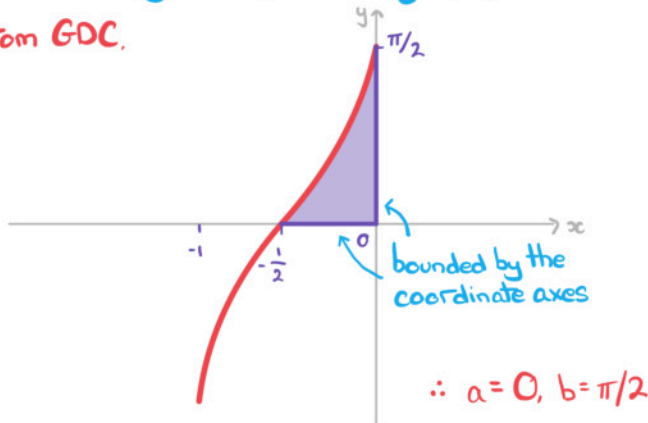
Your notes



Your notes

STEP 1: Identify limits, sketch graph/use GDC

From GDC,



STEP 2: Rearrange  $y=f(x)$  into  $x=g(y)$

$$y = \arcsin(2x+1)$$

$$\sin y = 2x+1$$

$$x = \frac{1}{2}(\sin y - 1)$$

STEP 3: Square  $x$

$$x^2 = \frac{1}{4}(\sin y - 1)^2$$

STEP 4: Find the volume

$$V = \pi \int_0^{\pi/2} \frac{1}{4}(\sin y - 1)^2 dy$$

As this is awkward, use your GDC but

- your GDC will expect the integrand in terms of  $x$
- remember  $\pi$  !

$$V = 0.279754 \dots$$

$$\therefore V = 0.280 \text{ cubic units (3 s.f.)}$$

## 5.9.5 Modelling with Volumes of Revolution

The volume of the solid of revolution formed by rotating an area through  $2\pi$  radians around the  $x$ -axis is

$V = \pi \int_a^b y^2 dx$ , and for the  $y$ -axis is  $V = \pi \int_a^b x^2 dy$ . These are both given in the **formula booklet**.



Your notes



Your notes

## Adding & Subtracting Volumes

### When would volumes of revolution need to be added or subtracted?

- The 'curve' boundary of an area may consist of **more than one** function of  $X$ 
  - For example
    - the 'curve' boundary from  $x = 0$  to  $x = 3$  is  $y = f(x)$
    - the 'curve' boundary from  $x = 3$  to  $x = 6$  is  $y = g(x)$
  - So the **total volume** would be  $V = \pi \int_0^3 [f(x)]^2 dx + \pi \int_3^6 [g(x)]^2 dx$
- The solid of revolution may have a 'hole' in it
  - e.g. a 'toilet roll' shape would be the **difference** of two cylindrical volumes

### How do I know whether to add or subtract volumes of revolution?

- When the **area** to be **rotated** around the  $X$ -axis has more than one function defining its boundary it can be trickier to tell whether to **add** or **subtract volumes of revolution**
  - It will depend on the **nature** of the **functions** and their **points of intersection**
  - With help from a GDC, sketch the graph of the functions and highlight the area required

### How do I solve problems involving adding or subtracting volumes of revolution?

- Visualising the solid created becomes increasingly useful (but also trickier) for shapes generated by separate volumes of revolution

- Continue trying to sketch the functions and their solids of revolution to help

#### STEP 1

Identify the functions ( $y = f(x)$ ,  $y = g(x)$ , ...) involved in generating the volume

Determine whether the separate volumes will need to be added or subtracted

Identify the limits for each volume involved

Sketching the graphs of  $y = f(x)$  and  $y = g(x)$ , or using a GDC to do so, is helpful, especially when the limits are not given directly in the question

#### STEP 2

Square  $y$  for all functions ( $[f(x)]^2$ ,  $[g(x)]^2$ , ...)

This step is not essential if a GDC can be used to calculate integrals and an exact answer is not required.

#### STEP 3

Use the appropriate volume of revolution formula for each part, evaluate the definite integral and add or subtract as necessary

The answer may be required in exact form

### Examiner Tip

- A sketch of the graph, limits, etc is always helpful, whether one has been given in the question or not
  - Use your GDC where possible



Your notes





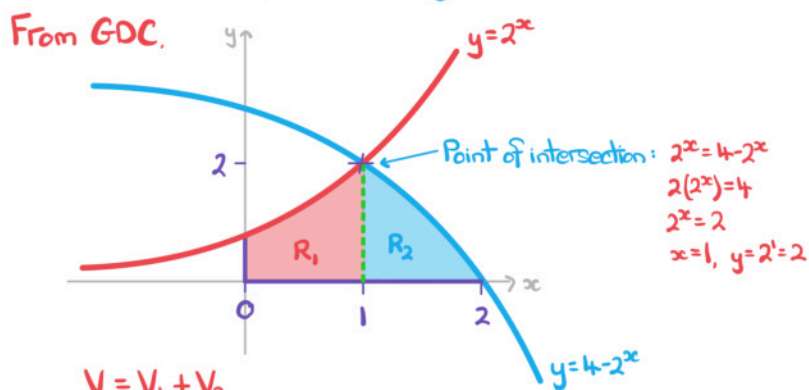
Your notes

### Worked example

Find the volume of revolution of the solid formed by rotating the region enclosed by the positive coordinate axes and the graphs of  $y = 2^x$  and  $y = 4 - 2^x$  by  $2\pi$  radians around the  $x$ -axis. Give your answer to three significant figures.

STEP 1: Identify functions, limits and whether to add or subtract

Use GDC to help sketch the graphs



$$V = V_1 + V_2$$

$$\text{For } R_1, a=0, b=1$$

$$\text{For } R_2, a=1, b=2$$

STEP 2: Square all functions - this step is not required in this question

STEP 3: Use formula for each part, evaluate and add

$$V = \pi \int_0^1 (2^x)^2 dx + \pi \int_1^2 (4 - 2^x)^2 dx$$

Use your GDC to evaluate - to avoid typing errors evaluate each integral separately, store in memory, then add

$$V = 6.798\ 540\dots + 4.941\ 881\dots = 11.740\dots$$

$$\therefore V = 11.7 \text{ cubic units (3 s.f.)}$$



Your notes

## Modelling with Volumes of Revolution

### What is meant by modelling volumes of revolution?

- Many everyday objects such as buckets, beakers, vases and lamp shades can be modelled as a **solid of revolution**
- The volume of revolution of the solid can then be calculated
- An object that would usually stand **upright** can be **modelled horizontally** so its **volume of revolution** can be found

### What modelling assumptions are there with volumes of revolution?

- The solids formed are usually the main shape of the body of the object
  - For example, the handle on a bucket would not be included
- The thickness of the solid is negligible relative to the size of the object
  - thickness will depend on the purpose of the object and the material it is made from

### How do I solve modelling problems with volumes of revolution?

- Visualising and sketching the solid formed can help with starting problems
- Familiarity with applying the volume of revolution formulae
  - around the x-axis:  $V = \int_a^b y^2 dx$
  - around the y-axis:  $V = \int_a^b x^2 dy$
- The volume of revolution may involve adding or subtracting partial volumes
- Questions may ask related questions in context
  - g. A question about a bucket may ask about its **capacity**
    - this would be measured in litres
    - so a conversion of units may be required
    - (1000 cm<sup>3</sup> = 1 litre)

### Examiner Tip

- Remember to answer questions directly
  - In modelling scenarios, interpretation is often needed after finding the 'final answer'
- Modelling questions often ask about assumptions, criticisms and/or improvements
- Examples
  - it is assumed the thickness of the material an object is made from is negligible
  - a 'smooth' curve may not be a good model if the item is being made from a rough material
  - other things may significantly reduce the volume found and impact conclusions
    - e.g. Stones, plants and decorations placed in an aquarium will reduce the volume of water needed to fill it - and hence the number/size/type of fish it can accommodate may be impacted



Your notes

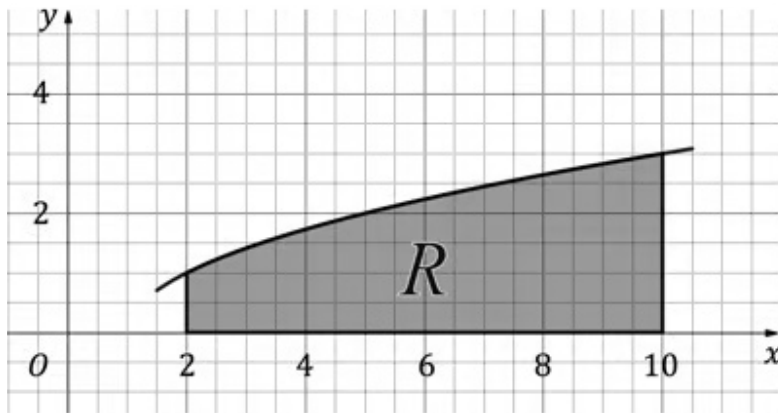


Your notes

 **Worked example**

The diagram below shows the region  $R$ , which is bounded by the function  $y = \sqrt{x - 1}$ , the lines  $x = 2$  and  $x = 10$ , and the  $x$ -axis.

Dimensions are in centimetres.



A mathematical model for a miniature vase is produced by rotating the region  $R$  through  $2\pi$  radians around the  $x$ -axis.

Find the volume of the miniature vase, giving your answer in litres to three significant figures.



Your notes

STEP 1 Identify limits

$$a=2$$

$$b=10$$

STEP 2 Square  $y$

$$y^2 = (\sqrt{x-1})^2 = x-1$$

STEP 3 Evaluate the integral

$$\begin{aligned} V &= \pi \int_2^{10} (x-1) dx = \pi [0.5x^2 - x]_2^{10} \\ &= \pi [(50-10) - (2-2)] \\ &= 40\pi \end{aligned}$$

Now we need to interpret this in the context of the miniature vase

$$V = 40\pi \text{ cm}^3$$

$$V = \frac{40\pi}{1000} \text{ litres}$$

$$1000 \text{ cm}^3 = 1 \text{ litre}$$

$$V = 0.125663 \dots$$

Volume of the miniature vase is 0.126 litres (3 s.f.)